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Generalized Solution of the Heat and Mass Transfer Problem

A.D. Nakhman*, Yu.V. Rodionov

Tambov State Technical University
112, Michurinskaya ul, Tambov, 392932, Russia

* Corresponding author. Tel.: +7 (4752) 63 04 59, +7 (4752) 63 18 01. *E-mail*: alextmb@mail.ru

Abstract

A universal mathematical model of heat and mass transfer processes (thermal conductivity in a finite rod, diffusion process in a finite hollow tube, stationary heat distribution in a half-plane) is constructed. This model is represented by a family of exponential means of the Fourier series of a periodic function. The concept of the initial condition in the case of nonstationary processes and the concept of the boundary condition in the case of a stationary heat distribution are specified. In this connection we establish the convergence of the means generated by a given periodic function at each of its Lebesgue points (in particular, at points of continuity) and at points of discontinuity of the first kind. Estimates of the rate of convergence are also proposed. The concept of a generalized Dirichlet problem is introduced and it is established that the corresponding family of exponential means is its solution.

Keywords

Universal model; heat and mass transfer problem; exponential means.

Introduction

Among the mathematical models of various objects, processes and phenomena, a special place is occupied by universal models. The universality property of mathematical models is manifested in the possibility of applying the same model to objects (systems) of fundamentally different nature, which obey different fundamental laws. The universality of mathematical models is explained, on the one hand, by the unity of the manifestation of the physical properties of the surrounding world, and by the abstractness of mathematical theories, their abstraction from the object of investigation on the other hand. The general theory of mathematical models, and in particular, the concept of universality is presented in [1], see also, e.g., [2, pp. 20–22]; [4].

In this paper we study the behavior of the family of the so-called exponential means of the Fourier series of a periodic function; a general theory of such means is presented in [4], [5]. This family in particular cases generates a universal mathematical model of heat and mass transfer processes (thermal conductivity in a finite rod, diffusion in a finite hollow tube, stationary heat distribution in a half-plane), see, e.g., [6, pp.145–151, 216–220]. The concept of the initial condition in the above nonstationary processes and the notion of the boundary condition in the generalized Dirichlet problem is refined in terms of the convergence of such means. Various types of convergence are studied and estimates of the rate of convergence are proposed.

The problem of studying of exponential means of Fourier series arises as a natural generalization of the classical problem of the limiting behavior of the Poisson integral in the sense of convergence almost everywhere and in the metric of Lebesgue spaces $L^p(p \geq 1)$; see [7, pp. 160–165]; [8, pp. 115–122]. One of the first results in this direction is contained in [9].

In the recent papers [10–12], a two-sided connection has been established between the problem of the convergence of means in the metrics of spaces of continuous functions, Lebesgue spaces $L^p(p \geq 1)$, and Lebesgue points with the problems of absolute convergence of the Fourier series and the integrability of Fourier transform of a “summing” exponential function. The results of [10–12] thereby actualize the problem of effective conditions on the "summing" sequence, which ensure the convergence of the corresponding generalized means of the Fourier series, and, in particular, the exponential means; such conditions were obtained in [5].

1. Statement of the problem

First, let us consider the problem of thermal conductivity in an isotropic and homogeneous rod of finite length with a constant coefficient of thermal diffusivity a^2

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}; \quad (1)$$

$$U(0,t) = U(\pi,t) = 0; \quad (2)$$

$$U(x,0) = f(x). \quad (3)$$

Here, $U = U(x,t)$ is the temperature at the point of the rod with abscissa x at time t; without loss of generality, the length of the rod is chosen to be equal π and the boundary conditions in (2) are zero. In the general case, we assume that the function $f = f(x)$ is 2π-periodic and is integrable by Lebesgue over the interval $(-\pi, \pi]$; in the context of problem (1) – (3) $f(x)$ is odd and represents the initial temperature distribution at the points of the rod.

The solution of problem (1) – (3) by the Fourier method (see, for example, [8, pp. 90–94]) leads to consideration of the sum of the series

$$U(f) = U(f,x;t) = \sum_{k=1}^{\infty} \exp(-ta^2k^2) b_k(f) \sin kx, \quad (4)$$

where $\{b_k(f)\}$ is the sequence of the sine-Fourier coefficients of the function f

$$b_k(f) = \frac{2}{\pi} \int_0^{\pi} f(t) \sin kt \, dt, \quad k = 1, 2, ...$$

The problem (1) – (3) also arises in the mathematical modeling of the process of diffusion of matter in a hollow straight tube of length equal to π. In this case, $U(f, x; t)$ is a concentration of the diffusing substance at the point x (at all points of the cross-section with the abscissa of x) at time t; here $a^2 = D/c$ under the assumption that the diffusion coefficient D and porosity coefficient c are constant; for definiteness, we assume that at the tube ends the concentration of matter is zero, and its distribution at the initial instant of time is given by the function $f(x)$. Now the sum of the series (4) is also a solution of the diffusion problem.

Finally, we consider the problem of finding a stationary temperature distribution $U = U(x, y)$ at points (x, y), $y > 0$ of a half-plane with a given temperature $f = f(x)$ at the boundary; this problem is known as the Dirichlet problem in the half-plane (see, for example, [8, pp. 150–152]).

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0; \quad (5)$$

$$U(f, x; 0) = f(x); \quad (6)$$

Its solution is the sum of the series

$$U(f) = U(f, x; y) = $$

$$= \sum_{k=-\infty}^{\infty} \exp(-y|k|) c_k(f) \exp(ikx), \quad h > 0, \quad (7)$$

where $\{c_k(f)\}$ there is a sequence of complex Fourier coefficients

$$c_k(f) = \frac{2}{\pi} \int_{-\pi}^{\pi} f(t) \exp(-ikt) \, dt, \quad k = 0, \pm 1, \pm 2, ...$$

Thus, the universal mathematical model of the processes considered above is a family of operators of the form $f \mapsto U_h(f)$, where

$$U_h(f) = U(f, x; \alpha; h) =$$

$$= \sum_{k=-\infty}^{\infty} \exp(-h|k|^{\alpha}) c_k(f) \exp(ikx), \quad h > 0, \quad \alpha > 0. \quad (8)$$

The case of $\alpha = 1$ generates the solution (7) of problem (5) – (6), and case $\alpha = 2$ – the solution (4) of problem (1) – (3).

One of the examples of the universal model being studied is the mathematical model of the drying process of plant material (however, modeling can be extended to the case of any capillary-porous material, including a number of polymers).

In the process of functioning of a two-stage vacuum-pulse dryer, the first stage is convective, where the air is the heat carrier. This period is characterized by a constant temperature throughout the volume. After removal of all surface moisture (about 50 %), the material begins to warm up. The distribution of temperature inside the material is of interest. If on the external surfaces the same constant temperature U_0 is maintained, and the initial temperature of the material is $f(x) = C_0$, $C_0 < V_0$, then we have the problem

$$\frac{\partial V}{\partial t} = a^2 \frac{\partial^2 V}{\partial x^2};$$

$$V(0,t) = V(\pi,t) = 0;$$

$$V(x,0) = C_0 - U_0,$$

where $V(x,t) = U(x,t) - U_0$.

According to (4), the temperature at each point x at each instant of time t is then determined in the form

$$U(x;t) = U_0 + \frac{4(C_0 - U_0)}{\pi};$$

$$\sum_{n=1}^{\infty} \exp(-ta^2(2n-1)^2) \frac{\sin(2n-1)x}{2n-1}.$$

This simplified model of the drying process demonstrates the effectiveness of exponential means of the Fourier series as an analytical apparatus for studying the heat and mass transfer process. A similar problem (from the point of view of the possibilities of using the universal model (8)) is mathematical modeling of the extraction process by means of a vacuum-evaporation extraction unit.

The formal substitution of $h = 0$ in (8) leads to the consideration of the Fourier series, which in the general case can be divergent (divergence at specific points or even everywhere). In this connection, it is necessary to clarify how we understand the condition $U(f,x;\alpha;0) = f(x)$, and, in particular, the conditions (3) and (6). The most natural here is the consideration $U(f,x;\alpha;0)$ as the limit of the form

$$\lim_{h\to+0} U(f,x;\alpha;h).$$

Thus, the following series of problems arise: to study the behavior of $U(f,x;\alpha;h)$ at $h \to +0$:

1) at each Lebesgue point of the function f (for the definition, see below in §2) and, in particular, at each of its points of continuity;

2) at any point of discontinuity of the first kind.

In doing so, we also solve:

3) the problem of obtaining at each point x an estimate of the deviation $|U(f,x;\alpha;h) - f(x)|$.

2. Main results

Let $f = f(x)$ be an arbitrary 2π-periodic function from the space $L^p_{2\pi}$ of all summable with the p-th power on $[-\pi,\pi]$ functions ($p \ge 1$, $L_{2\pi} = L^1_{2\pi}$).

Let $C_{2\pi}$ also be the space of 2π-periodic continuous functions with the norm

$$\| f \|_{C_{2\pi}} = \max_{x\in[-\pi,\pi]} | f(x) |$$

A point x is called Lebesgue point of a function $f \in L_{2\pi}$ if it has the property

$$\int_{-\eta}^{\eta} | f(x+t) - f(x) | \, dt = o(\eta), \; \eta \to +0.$$

We note that the Lebesgue points of the functions $f \in L_{2\pi}$ are located almost everywhere [9, p. 111]. It is obvious that every point x of continuity of a function $f(x)$ is its Lebesgue point.

The following theorem proposes the solution of problem 3).

Theorem 1. For each one $\alpha > 0$ we have the estimate

$$\left|U(f,x;\alpha;h) - f(x)\right| \le C_\alpha \sum_{k=0}^{\infty} (k+1) \times$$

$$\times \left|\exp(-hk^\alpha) - 2\exp\left(-h(k+1)^\alpha\right) + \exp\left(-h(k+2)^\alpha\right)\right| \times$$

$$\times \left(\sup_{j=0,1,\dots;\, j<\log_2(2\pi k)} \frac{k}{2^j} \int_{-2^j/k}^{2^j/k} \left| f(x+t) - f(x)\right| dt \right).$$

In particular,

$$\sup_{h>0} \left|U(f,x;\alpha;h) - f(x)\right| \le$$

$$\le C_\alpha \sup_{\eta>0} \frac{1}{\eta} \int_{-\eta}^{\eta} \left| f(x+t) - f(x)\right| dt. \qquad (9)$$

In the following assertion, the convergence of processes (8) is established, so that the result of Theorem 1 is now an estimate of the rate of convergence.

Theorem 2.

1) The relation

$$\lim_{h\to+0} U(f,x;\alpha;h) = f(x) \qquad (10)$$

is valid at every Lebesgue point of function $f \in L_{2\pi}$ for all $\alpha > 0$.

In particular, the relation (10) holds at each point x of continuity of function f and uniformly in x for every continuous 2π-periodic $f(x)$.

2) Suppose that both unilateral limits $f(x-0)$ and $f(x+0)$ exist and are finite at the point x. Then for each $\alpha > 0$ we have the relation

$$\lim_{h\to+0} U(f,x;\alpha;h) = \frac{f(x-0) + f(x+0)}{2}. \qquad (11)$$

3. Supporting statements

We consider the first and second finite differences of the sequence

$$\{\exp(-hk^\alpha)\}, \quad k = 0,1,\dots: \qquad (12)$$

$$\Delta_k = \Delta_k(h,\alpha) = \exp(-hk^\alpha) - \exp(-h(k+1)^\alpha)$$

and

$$\Delta_k^2 = \Delta_k^2(h,\alpha) = \Delta_k(h,\alpha) - \Delta_{k+1}(h,\alpha).$$

Lemma 1. The sequence (12) is convex for each $h > 0$, if $0 < \alpha \le 1$ and is piecewise convex if $\alpha > 1$.

Indeed, by the Lagrange theorem, for some $\theta_1, \theta_2 \in (0,1)$ and $\theta \in (0,2)$, generally speaking, depending on the chosen *k*, we obtain

$$\Delta_k(h,\alpha) = \alpha h (k+\theta_1)^{\alpha-1} \exp\left(-h(k+\theta_1)^{\alpha}\right),$$

$$\Delta_k^2(h,\alpha) = (1+\theta_1)\alpha h\left(\alpha h (k+\theta)^{\alpha} - (\alpha-1)\right)\times$$
$$\times (k+\theta)^{\alpha-2}\exp(-h(k+\theta)^{\alpha}). \qquad (13)$$

According to (13), we have $\Delta_k^2(h,\alpha) \ge 0$ for all values of *k* and for $0 < \alpha \le 1$. If $\alpha > 1$, then the second differences change the sign "at a point" $\left(\frac{\alpha-1}{\alpha h}\right)^{1/\alpha} - \theta.$ Given the fact that *k* is an integer, we have: $\Delta_k^2(h,\alpha)$ preserve the minus sign for integer values of *k* (depending on h), smaller $\left(\frac{\alpha-1}{\alpha h}\right)^{1/\alpha} - 2,$ and the plus sign for *k* greater $\left(\frac{\alpha-1}{\alpha h}\right)^{1/\alpha}$. Thus, the sequence (12) is piecewise convex if $\alpha > 1$. Lemma 1 is proved.

Put

$$R_k(f,x) = \sup_{j=0,1,\dots;\, j<\log_2(2\pi k)} \frac{k}{2^j}\int_{-2^j/k}^{2^j/k} |f(x+t)-f(x)|\,dt. \qquad (14)$$

According to the definition of Lebesgue points, for any $\varepsilon > 0$ at such a point *x* we have the inequality

$$R_k(f,x) < \varepsilon,$$

if the values $\eta = 2^j/k$ are sufficiently small. Further, by the summability of *f*, $R_k(f,x)$ is finite for the remaining values $2^j/k,\ k = 1,2,\dots$

We introduce the so-called Dirichlet kernels and the Fejer kernel [7, p. 86, 148]

$$D_k(t) = \frac{1}{2} + \sum_{\nu=1}^{k}\cos\nu t = \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{1}{2}t} \quad \text{and}$$

$$F_k(t) = \frac{1}{k+1}\sum_{\nu=0}^{k} D_k(t) = \frac{\sin^2\frac{k+1}{2}t}{2(k+1)\sin^2\frac{1}{2}t} \qquad (15)$$

respectively.

Lemma 2. The following estimates hold:

$$\int_{-\pi}^{\pi} |f(x+t)-f(x)|\cdot|D_k(t)|\,dt \le$$
$$\le C R_{k+1}(f,x)\ln(k+1), \quad k = 1,2,\dots, \qquad (16)$$

$$\int_{-\pi}^{\pi} |f(x+t)-f(x)| F_k(t)\,dt \le C R_{k+1}(f,x), \quad k = 1,2,\dots \qquad (17)$$

Proof. Denote $\varphi_x(t) = f(x+t) - f(x)$ and use the obvious for (15) inequalities

$$|D_k(t)| + F_k(t) \le C(k+1), \quad k = 0,1,\dots, \qquad (18)$$

$$|D_k(t)| \le C\frac{1}{|t|}, \quad 0 < |t| \le \pi, \quad k = 0,1,\dots, \qquad (19)$$

$$|F_k(t)| \le C\frac{1}{(k+1)t^2}, \quad 0 < |t| \le \pi, \quad k = 0,1,\dots, \qquad (20)$$

Let us establish (16). According to (14), (18), (19), we have

$$\int_{-\pi}^{\pi} |\varphi_x(t)|\cdot|D_k(t)|\,dt \le$$

$$\le C\Big((k+1)\int_{|t|\le\frac{1}{k+1}} |\varphi_x(t)|\,dt + \sum_{j=1}^{S}\frac{k+1}{2^{j-1}}\int_{\frac{2^{j-1}}{k+1}\le t\le\frac{2^j}{k+1}} |\varphi_x(t)|\,dt\Big) \le$$

$$\le C(1+2S)R_{k+1}(f,x) \le CR_{k+1}(f,x)\ln(k+1),$$

and for each $k = 0,1,\dots$ choose a natural number *S* such that $\frac{2^{S-1}}{k+1} \le \pi < \frac{2^S}{k+1}.$

Further, we prove (17). By virtue of (18) and (20), we obtain

$$\int_{-\pi}^{\pi} |\varphi_x(t)|\,F_k(t)\,dt \le C\Big((k+1)\int_{|t|\le\frac{1}{k+1}} |\varphi_x(t)|\,dt +$$
$$+\sum_{j=1}^{S}\frac{k+1}{(2^{j-1})^2}\int_{\frac{2^{j-1}}{k+1}\le t\le\frac{2^j}{k+1}} |\varphi_x(t)|\,dt\Big) \le CR_{k+1}(f,x).$$

The lemma is proved.

4. Proof of Theorem 1

Using the integral form of the complex Fourier coefficients $c_k(f)$ in the notation (8), we obtain by the assistance of the Abel transform [7, p. 15]

$$U(f,x;\alpha;h)-f(x)=\lim_{N\to+\infty}\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(t)-f(x)\right)\times$$

$$\times\left\{\frac{1}{2}+\sum_{k=1}^{N}\exp(-hk^{\alpha})\cos k(x-t)\right\}dt=$$

$$=\frac{1}{\pi}\lim_{N\to+\infty}\left\{\exp(-hN^{\alpha})\int_{-\pi}^{\pi}\varphi_x(t)D_N(t)\,dt+\right.$$

$$\left.+\int_{-\pi}^{\pi}\varphi_x(t)\sum_{k=1}^{N-1}\Delta_k(h,\alpha)D_k(t)dt\right\}.$$

Applying the obvious relation

$$D_k(t)=(k+1)F_k(t)-kF_{k-1}(t),\quad k=0,1,...$$

and (again) the Abel transform, we obtain

$$U(f,x;\alpha;h)-f(x)=$$

$$=\frac{1}{\pi}\lim_{N\to+\infty}\left\{\exp\left(-hN^{\alpha}\right)\int_{-\pi}^{\pi}\varphi_x(t)D_N(t)\,dt+\right.$$

$$+N\Delta_k(h,\alpha)\int_{-\pi}^{\pi}\varphi_x(t)F_{N-1}(t)\,dt+$$

$$\left.+\sum_{k=0}^{N-2}(k+1)\Delta_k^2(h,\alpha)\int_{-\pi}^{\pi}\varphi_x(t)F_k(t)\,dt\right\}.\qquad(21)$$

By estimates (16) and (17), we have

$$\int_{-\pi}^{\pi}|\varphi_x(t)|\cdot|D_N(t)|dt+N|\Delta_N(h,\alpha)|\int_{-\pi}^{\pi}|\varphi_x(t)|F_{N-1}(t)\,dt\le$$

$$\le C\left(R_{N+1}(f,x)+R_N(f,x)\right)\left\{\exp(-hN^{\alpha})\ln(N+1)+\right.$$

$$\left.+\alpha hN\exp(-hN^{\alpha})N^{\alpha-1}\right\}.$$

Obviously, here the relations

$$\exp(-hN^{\alpha})\ \ln(N+1)\to 0$$

and

$$\alpha hN^{\alpha}\exp(-hN^{\alpha})\to 0\quad\text{for}\quad N\to\infty\qquad(22)$$

hold, as is easily verified by the L'Hospital's rule. Hence, according to (21),

$$U(f,x;\alpha;h)-f(x)=\sum_{k=0}^{\infty}(k+1)\Delta_k^2(h,\alpha)\int_{-\pi}^{\pi}\varphi_x(t)F_k(t)\,dt,\qquad(23)$$

and, consequently (see (17)),

$$\left|U(f,x;\alpha;h)-f(x)\right|\le C_{\alpha}\sum_{k=0}^{\infty}(k+1)\left|\Delta_k^2(h,\alpha)\right|R_{k+1}(f,x).\qquad(24)$$

Since

$$R_k(f,x)\le\sup_{\eta>0}\frac{1}{\eta}\int_{-\eta}^{\eta}|f(x+t)-f(x)|\,dt,$$

then

$$\left|U(f,x;\alpha;h)-f(x)\right|\le$$

$$\le C_{\alpha}\left(\sup_{\eta>0}\frac{1}{\eta}\int_{-\eta}^{\eta}|f(x+t)-f(x)|\,dt\right)\sum_{k=0}^{\infty}(k+1)\left|\Delta_k^2(h,\alpha)\right|.\qquad(25)$$

As it was stated in Lemma 1 (see (13)), for all k and $0<\alpha\le 1$ the values $\Delta_k^2(h,\alpha)\ge 0$. In this case, according to the Abel transform we have

$$\sum_{k=0}^{N}(k+1)\Delta_k^2(h,\alpha)=\sum_{k=0}^{N}\Delta_k(h,\alpha)-(N+1)\Delta_{N+1}(h,\alpha).\qquad(26)$$

Passing to limit in (26) and taking into account (22), we get this limit to be unity. Thus, the series

$$\sum_{k=0}^{\infty}(k+1)\left|\Delta_k^2(h,\alpha)\right|\qquad(27)$$

in (25) is convergent and so the assertion of Theorem 1 for $0<\alpha\le 1$ is proved.

To prove the convergence of the series (27) in order to $\alpha>1$, we use the result of Lemma 1 on the piecewise convexity of the sequence (12), namely, we apply the transformation (26) for *N*, which is equal to the integer part of the number

$$\left(\frac{\alpha-1}{\alpha h}\right)^{1/\alpha}-2$$

and the transform

$$\sum_{k=N+1}^{\infty}(k+1)\Delta^2{}_k(h,\alpha)=$$

$$=\lim_{n\to\infty}\left(\exp(-h(N+2)^{\alpha})-\exp(-h(n+1)^{\alpha})+\right.$$

$$\left.+(N+2)\Delta_{N+1}(h,\alpha)-(n+1)\Delta_{n+1}(h,\alpha)\right)$$

for *N*, which are greater the integer part of $\left(\frac{\alpha-1}{\alpha h}\right)^{1/\alpha}$ (see the proof of Lemma 1).

It remains to use the second of the relations (22), and, in particular, the uniform boundedness (the boundedness by a constant depending only on α) of the sequence $\left\{\alpha hk^{\alpha}\exp(-hk^{\alpha})\right\}$.

According to (24), the estimate (9) is now established for all $\alpha>0$, and Theorem 1 is proved.

Remark. The estimate (24), which was the basis of the proof, also has an independent interest. Namely, as the reasoning shows, it remains valid not only for the sequence (12), but for every summing sequence $\{\tau_k\},\ k=0,1,...,$ that has the properties

$$\tau_k(h)=o\left(\frac{1}{\ln k}\right) \quad \text{and}$$

$$\Delta_k\tau_k(h)=o\left(\frac{1}{k}\right), \quad k\to\infty.$$

5. Proof of Theorem 2

According to the classical results [7, p.113, 151], for any $\varepsilon>0$ at every Lebesgue point, the relation

$$\int_{-\pi}^{\pi}|\varphi_x(t)|F_k(t)\,dt\le C\varepsilon$$

holds for all values k, which are greater than some $\nu=\nu(\varepsilon,x)$; the constant C does not depend on ε and k. Taking into account (23) and (17), we have

$$|U(f,x;\alpha;h)-f(x)|\le C\left(\sup_{\eta>0}\frac{1}{\eta}\int_{-\eta}^{\eta}|f(x+t)-f(x)|\,dt\right)\times$$

$$\times\sum_{k=0}^{\nu-1}(k+1)\left|\Delta_k^2(h,\alpha)\right|+C\varepsilon\sum_{k=\nu+1}^{\infty}(k+1)\left|\Delta^2\lambda_k(h)\right|. \tag{28}$$

Further, according to (13), $\Delta^2\lambda_k(h)\to 0$ for $h\to+0$ and $k=0,1,...,\nu,$ so that the first of the sums (consisting of a fixed number of terms) written on the right-hand side of (28) tends to zero. At the same time, the second of the sums in (28) does not exceed (27), and, as it was shown in the proof of Theorem 1, this sum is uniformly bounded for all $\alpha>0$ (does not exceed some constant depending only on α). Hence, (28) implies the estimate

$$|U(f,x;\lambda,h)-f(x)|\le C\varepsilon,$$

from which (in view of the arbitrariness of ε) the validity of relation (10) follows at each Lebesgue point.

It is obvious that every point of continuity of a function is also a Lebesgue point, therefore (10) remains valid at points of continuity of the function f.

To prove uniform convergence, we note that transformations, analogous to (21), lead to the estimate

$$|U(f,x;\alpha;h)|\le$$

$$\le C\lim_{N\to+\infty}\left\{\exp(-hN^{\alpha})\int_{-\pi}^{\pi}|f(x+t)|\cdot|D_N(t)|\,dt+\right.$$

$$\left.+N\Delta_N(h,\alpha)\int_{-\pi}^{\pi}|f(x+t)|F_N(t)\,dt\right\}+$$

$$+\sum_{k=0}^{\infty}(k+1)\left|\Delta_k^2(h,\alpha)\right|\int_{-\pi}^{\pi}|f(x+t)|F_k(t)\,dt,$$

and, consequently, by virtue of (22), we obtain

$$|U(f,x;\alpha;h)|\le C\,\|f\|_{C_{2\pi}}\sum_{k=0}^{\infty}(k+1)\left|\Delta^2\lambda_k(h)\right|.$$

Thus, the family of norms $\|U_h\|$ of each of the operators: $U_h: f\mapsto U_h(f)$, acting from $\mathbf{C}_{2\pi}$ in $\mathbf{C}_{2\pi}$, is uniformly (with respect to h) bounded. It remains to note that

$$\lim_{h\to+0}\exp(-hk^{\alpha})=1 \tag{29}$$

and now the convergence uniformly with respect to x holds by the Banach–Steinhaus theorem.

To prove the second part of the theorem, we denote by

$$l(f,x)=\frac{f(x-0)+f(x+0)}{2}$$

and

$$\varphi_x(t)=f(t)-l(f,x).$$

In the same way as we have obtained (21), we have

$$U(f,x;\alpha;h)-l(f,x)=$$

$$=\frac{1}{\pi}\lim_{N\to+\infty}\left\{\exp(-hN^{\alpha})\int_{-\pi}^{\pi}\varphi_x(t)D_N(t)\,dt+\right.$$

$$+N\Delta_k(h,\alpha)\int_{-\pi}^{\pi}\varphi_x(t)F_{N-1}(t)\,dt+$$

$$\left.+\sum_{k=0}^{N-2}(k+1)\Delta_k^2(h,\alpha)\int_{-\pi}^{\pi}\varphi_x(t)F_k(t)\,dt\right\}. \tag{30}$$

Further, for every even 2π-periodic summable function $g(x)$ the equality

$$\int_{-\pi}^{\pi}\varphi_x(t)g(t)dt=\int_0^{\pi}\big(f(x+t)+f(x-t)-2l(f,x)\big)g(t)dt=$$

$$=\int_0^{\pi}\big(f(x+t)-f(x+0)\big)g(t)dt+$$

$$+\int_0^{\pi}\big(f(x-t)-f(x-0)\big)g(t)dt$$

holds.

It is obvious now that it suffices to consider the right-hand side of (30), in which the integration is carried out over $[0, \pi]$ and $\varphi_x(t)$ replaced by $f(x+t)-f(x+0)$; the case $f(x-t)-f(x-0)$ can be treated in a completely analogous way.

According to the condition of the second part of Theorem 2, the function $f(x)$ has right-hand continuity at a point x, and therefore for every $\varepsilon > 0$ there is such $\delta = \delta(\varepsilon) > 0$, that

$$|f(x+t)-f(x+0)| < \varepsilon \quad \text{for} \quad 0 < t < \delta.$$

Consequently, according to (18) and (19),

$$\int_0^\pi |f(x+t)-f(x+0)|\cdot|D_N(t)|dt < \int_0^\delta \varepsilon\cdot|D_N(t)|dt +$$
$$+\int_\delta^\pi |f(x+t)-f(x+0)|\frac{1}{t}dt <$$
$$< \varepsilon\int_0^\pi |D_N(t)|dt + \frac{1}{\delta}\int_\delta^\pi |f(x+t)-f(x+0)|dt <$$
$$< C\left(\varepsilon\ln N + \frac{1}{\delta}\int_{-\pi}^{\pi}|f(t)|dt + \frac{\pi}{\delta}|f(x+0)|\right).$$

If the right-hand side of the last relation is multiplied by $\exp(-hN^\alpha)$, then the product will tend to zero for $N\to\infty$.

In a similar way, taking into account (20), we obtain

$$\int_0^\pi |f(x+t)-f(x+0)| \ F_k(t)dt < \int_0^\delta \varepsilon F_k(t)dt +$$
$$+\int_\delta^\pi |f(x+t)-f(x+0)|\frac{1}{(k+1)t^2}dt <$$
$$< \varepsilon\int_0^\pi F_k(t)dt + \frac{1}{\delta(k+1)}\int_\delta^\pi |f(x+t)-f(x+0)|dt <$$
$$< C(\varepsilon + \frac{1}{\delta(k+1)}\int_{-\pi}^{\pi}|f(t)|dt + \frac{\pi}{\delta(k+1)}|f(x+0)|). \quad (31)$$

Taking into account (31) and the relation $N\Delta_k(h,\alpha)\to 0$ ($N\to\infty$), then (see (30)), it remains to estimate

$$\sum_{k=0}^{\infty}(k+1)|\Delta_k^2(h,\alpha)|\int_0^\pi |f(x+t)-f(x+0)| \ F_k(t)dt.$$

This sum, by (31), does not exceed

$$C\left\{\varepsilon\sum_{k=0}^{\infty}(k+1)|\Delta_k^2(h,\alpha)| + \frac{\pi}{\delta}|f(x+0)|\sum_{k=0}^{\infty}|\Delta_k^2(h,\alpha)|\right\}. \quad (32)$$

It was shown above the sum (27) is finite. The second sum in (32) turns into two sums of sign-constant terms due to the piecewise convexity (12).

$$\sum_{k=0}^{\nu-1}\Delta_k^2(h,\alpha) = \Delta_0(h,\alpha)-\Delta_\nu(h,\alpha) \quad (33)$$

and

$$\sum_{k=\nu}^{\infty}\Delta_k^2(h,\alpha) = \Delta_\nu(h,\alpha). \quad (34)$$

If $0<\alpha<1$, then it is obvious that

$$\sum_{k=0}^{\infty}\Delta_k^2(h,\alpha) = \Delta_0(h,\alpha). \quad (35)$$

Now we note that each of the finite differences in the right-hand sides of (33) – (35) tends to zero for $h\to+0$, since (29) holds for all $k = 0, 1, \ldots$.

Returning to (32), we see that its right-hand side becomes smaller $C\varepsilon$, and, in view of arbitrariness of $\varepsilon>0$, the deviation modulus $U(f,x;\alpha;h)-l(f,x)$ in (30) is infinitesimal (when $h\to+0$) at each point x of discontinuity of the first kind.

The second part of Theorem 2 (the relation (11)) is completely proved.

6. Application to the solution of the generalized Dirichlet problem

The results of Theorem 2 can be applied to the solution of the following generalized Dirichlet problem in the half-plane (x,h), $h>0$. Let

$$\Delta_\alpha U = i^{2-2\alpha}\frac{\partial^{2\alpha}U}{\partial x^{2\alpha}} + \frac{\partial^2 U}{\partial h^2}, \qquad \alpha>0,$$

be the generalized Laplace operator; here the differentiation with respect to x is the corresponding fractional differentiation. It is required to find the solution $U = U(x;h)$ of the boundary value problem

$$\Delta_\alpha U = 0, \ U|_\Gamma = l, \quad (36)$$

where Γ is the boundary $h=0$ of the half-plane.

The boundary condition in (36) is understood as follows

$$\lim_{h\to+0}U(x;h) = l(x), \quad (37)$$

wherein for a given 2π-periodic summable function $f = f(x)$

$$l(x) = l(f,x) = f(x),$$

if x is the Lebesgue point (in particular, a point of continuity), and

$$l(f,x) = \frac{f(x-0)+f(x+0)}{2},$$

if x is a point of discontinuity of the first kind.

It is easy to verify that the sum $U(x;h) = U(f, x;\alpha;h)$ of (8) satisfies to the equation $\Delta_\alpha U = 0;$ in this case the possibility of termwise differentiation will be ensured by the convergence of the numerical series (majorant for series of corresponding partial derivatives)

$$\sum_{k=1}^{\infty}(k^{2\alpha}+k^2)\exp(-hk^\alpha). \qquad (38)$$

In turn, it is not difficult to show that (38) converges simultaneously with the improper integral

$$\int_0^\infty x^s \exp(-hx^\alpha)dx = C_h \int_0^\infty t^{\frac{s+1}{\alpha}-1} \exp(-t)dt,$$

representing the gamma function of a positive argument $\frac{s+1}{\alpha}$; here $s = 2\alpha$ or $s = 2$.

Finally, the boundary condition (37) in the above sense is satisfied according to the assertion of Theorem 2.

Conclusions

Let us formulate the main results of the paper. The family of exponential means $U(f, x;\alpha;h)$ of the Fourier series of function $f \in L_{2\pi}$ is a universal mathematical model of heat and mass transfer processes (the thermal conductivity in a finite rod, the diffusion process in a finite hollow tube, the stationary heat distribution in a half-plane). One of the examples of the universal model being studied is the mathematical model of the drying process of plant material.

The formal application of the Fourier method in heat and mass transfer problems can lead to the appearance of series that lack convergence. Because of this, there is a need to study the behavior of $U(f, x;\alpha;h)$ at $h \to +0$.

The following results are established. The relation

$$\lim_{h\to+0} U(f, x;\alpha;h) = f(x).$$

holds at every Lebesgue point of the function $f \in L_{2\pi}$ (and, in particular, at each of its points of continuity) for all $\alpha > 0$. At each point of the discontinuity of the first kind the family $U(f, x;\alpha;h)$ converges (at $h \to +0$) to the half-sum of the unilateral limits of the function $f(x)$.

The obtained limit relations are applied to the solution of the problem

$$\Delta_\alpha U = 0, \; U|_\Gamma = l,$$

where Γ is the boundary $h = 0$ of the half-plane, and $\Delta_\alpha U$ is the generalized Laplace operator.

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